Static Program Analysis

Part 5 – widening and narrowing

http://cs.au.dk/~amoeller/spa/

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Interval analysis

- Compute upper and lower bounds for integers
- Possible applications:
  - array bounds checking
  - integer representation
  - …
- Lattice of intervals:

$$Interval = lift(\{\ [l, h]\ \mid l,h \in N \wedge l \leq h\ \})$$

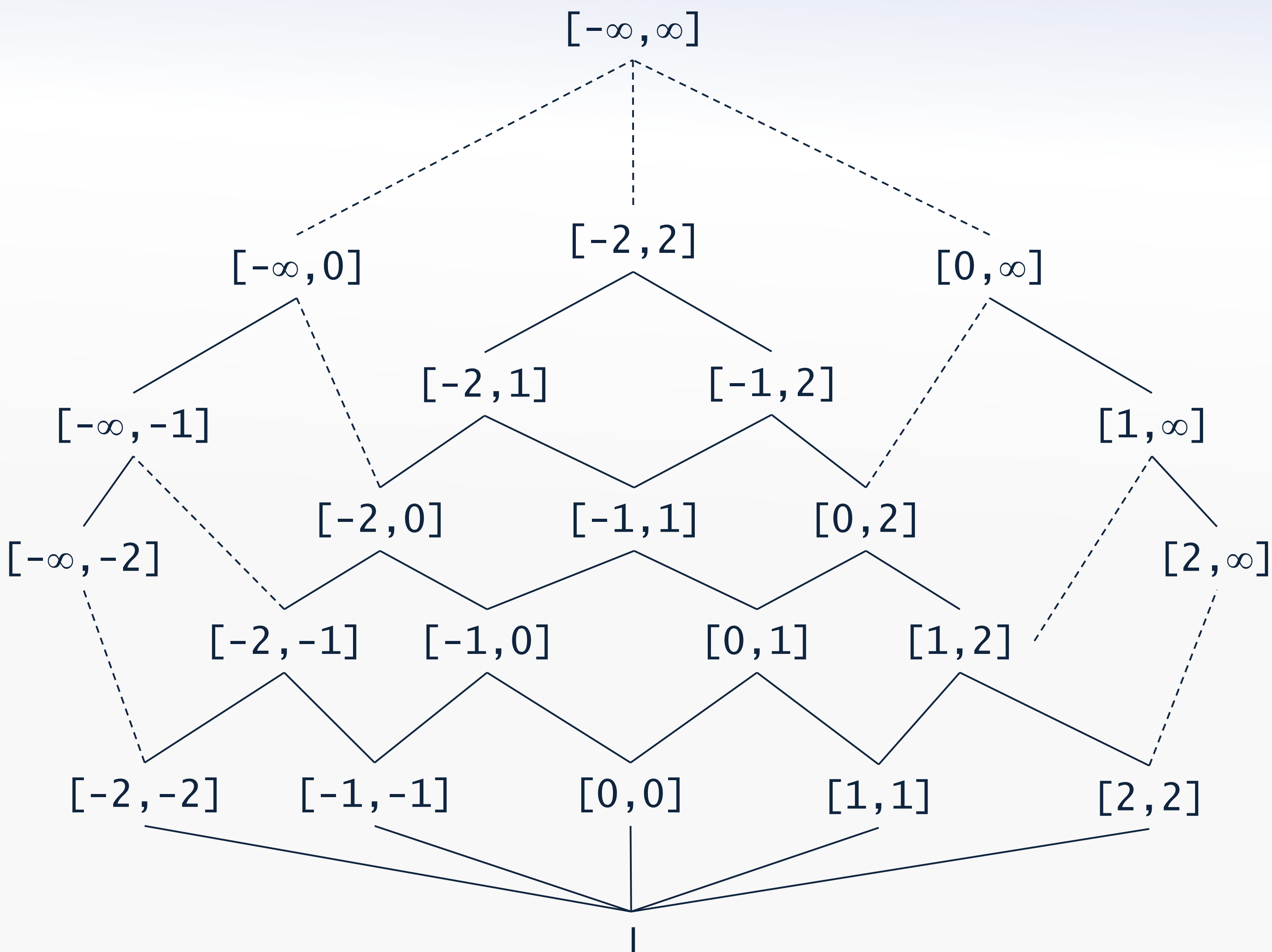
where

$$N = \{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$$

and intervals are ordered by inclusion:

$$[l_1, h_1] \sqsubseteq [l_2, h_2] \text{ iff } l_2 \leq l_1 \wedge h_1 \leq h_2$$

The interval lattice



bottom element here interpreted as “not an integer”

Interval analysis lattice

- The total lattice for a program point is

  $$L = \mathit{Vars} \rightarrow \mathit{Interval}$$

  that provides bounds for each (integer) variable

- If using the worklist solver that initializes the worklist with only the *entry* node, use the lattice *lift*(L)
  - bottom value of *lift*(L) represents "unreachable program point"
  - bottom value of L represents "maybe reachable, but all variables are non-integers"

- This lattice has *infinite height*, since the chain

  $$[0,0] \sqsubseteq [0,1] \sqsubseteq [0,2] \sqsubseteq [0,3] \sqsubseteq [0,4] \ldots$$

  occurs in *Interval*

Interval constraints

- For assignments:

$$[\![\, x = E \,]\!] = JOIN(\mathrm{v})[x \rightarrow eval(JOIN(\mathrm{v}), E)]$$

- For all other nodes:

$$[\![\mathrm{v}]\!] = JOIN(\mathrm{v})$$

where $JOIN(\mathrm{v}) = \bigsqcup_{\mathrm{w} \in pred(\mathrm{v})} [\![\mathrm{w}]\!]$

Evaluating intervals

- The *eval* function is an *abstract evaluation*:
  - $eval(\sigma, x) = \sigma(x)$
  - $eval(\sigma, intconst) = [intconst, intconst]$
  - $eval(\sigma, E_1 \text{ op } E_2) = \overline{\text{op}}(eval(\sigma,E_1), eval(\sigma,E_2))$
- Abstract arithmetic operators:
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) =$

not trivial to implement!

$$[\min_{x \in [l_1, h_1], y \in [l_2, h_2]} x \text{ op } y, \max_{x \in [l_1, h_1], y \in [l_2, h_2]} x \text{ op } y]$$

- Abstract comparison operators (could be improved):
  - $\overline{\text{op}}([l_1, h_1], [l_2, h_2]) = [0, 1]$

Fixed-point problems

- The lattice has infinite height, so the fixed-point algorithm does not work ☹

- In L^n, the sequence of approximants

  $f^i(\bot, \bot, \ldots, \bot)$

  is not guaranteed to converge
- (Exercise: give an example of a program where this happens)

- Restricting to 32 bit integers is not a practical solution
- *Widening* gives a useful solution…

Widening

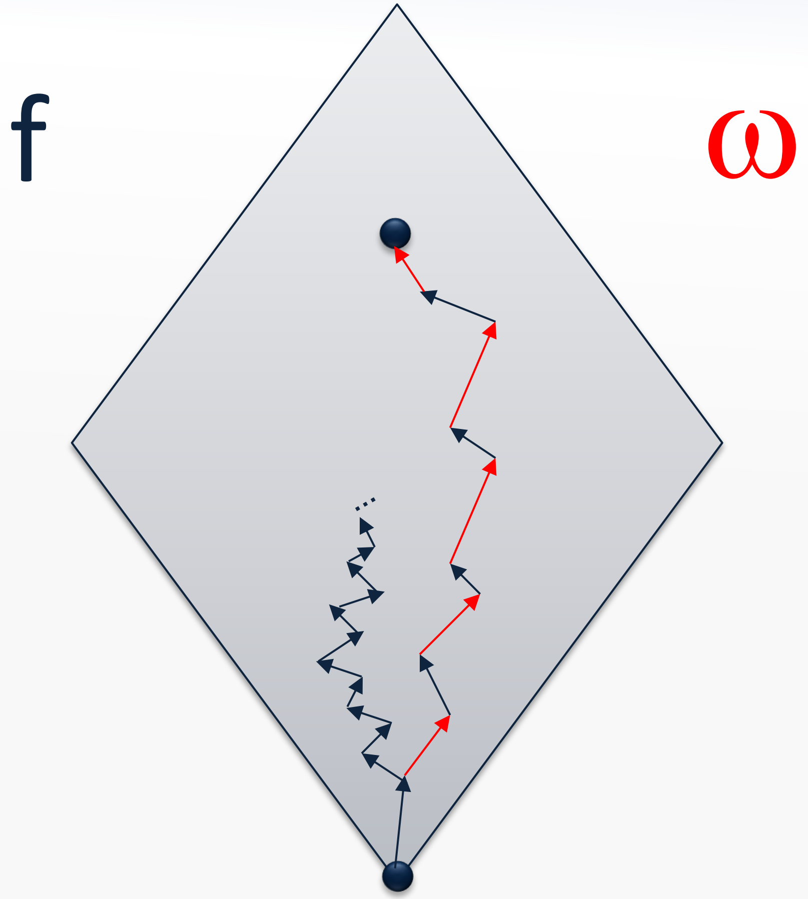
- Introduce a *widening* function ω: $L^n \rightarrow L^n$ so that

  $$(\omega \circ f)^i(\bot, \bot, ..., \bot)$$

  converges on a fixed-point that is a safe approximation of each $f^i(\bot, \bot, ..., \bot)$

- i.e. the function ω coarsens the information

Turbo charging the iterations



Widening for intervals

- The function ω is defined pointwise on L^n
- Parameterized with a fixed finite subset $B \subset N$
  - must contain $-\infty$ and ∞ (to retain the ⊤ element)
  - typically seeded with all integer constants occurring in the given program
- Idea: Find the nearest enclosing allowed interval
- On single elements from *Interval* :

$$\omega([a, b]) = [\ max\{i \in B \mid i \leq a\},\ min\{i \in B \mid b \leq i\}\ ]$$

$$\omega(\bot) = \bot$$

Divergence in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [8,8], y \to [0,1]]$
$[x \to [8,8], y \to [0,2]]$
$[x \to [8,8], y \to [0,3]]$
…

Widening in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of widening

- Widening works when:
  - ω is an *extensive* and *monotone* function, and
  - $\omega(L)$ is a *finite-height* lattice

- A few key concepts for widening
  - x is a fixpoint of f, i.e., $f(x) = x$
  - f is *extensive* at x iff $f(x) \sqsupseteq x$
  - f is *reductive* at x iff $f(x) \sqsubseteq x$

- Extensive functions never underestimate the fixpoint
- Reductive functions never overestimate the fixpoint

Correctness of widening

- Widening works when:
  - ω is an *extensive* and *monotone* function, and
  - $\omega(L)$ is a *finite-height* lattice
- Safety: $\forall i: f^i(\bot, \bot, ..., \bot) \sqsubseteq (\omega \circ f)^i(\bot, \bot, ..., \bot)$
  since f is monotone and ω is extensive
- $\omega \circ f$ is a monotone function $\omega(L) \rightarrow \omega(L)$
  so the fixed-point exists
- Almost “correct by definition”
- When used in the worklist algorithm, it suffices to apply widening on back-edges in the CFG

Narrowing

- Widening generally shoots over the target
- *Narrowing* may improve the result by applying f
- Define:

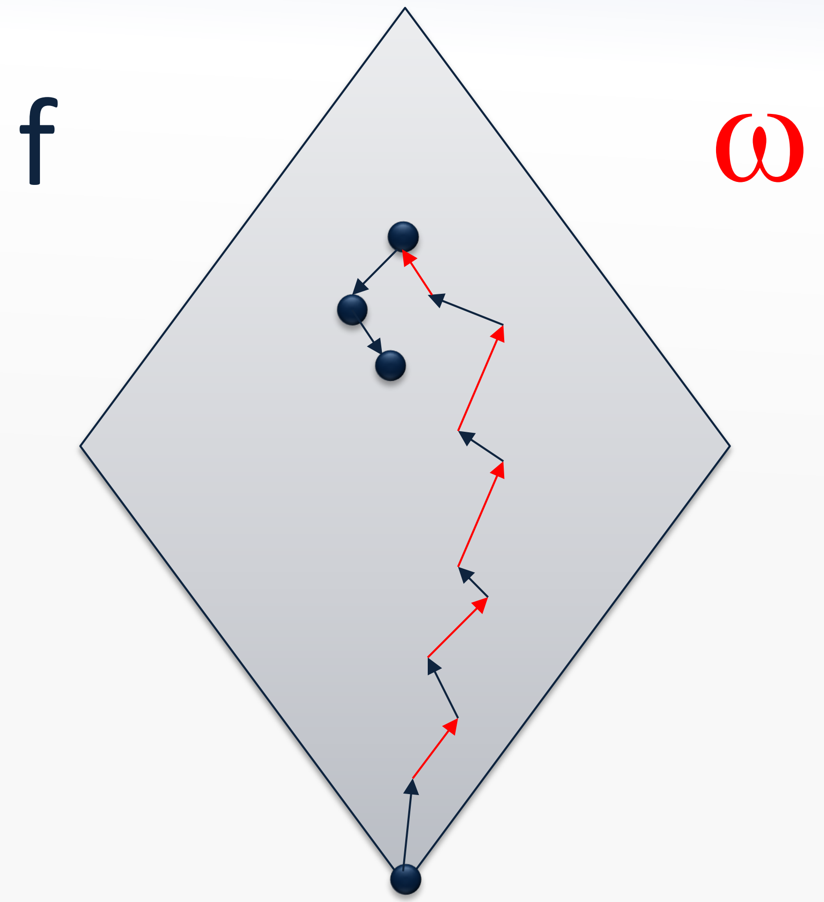
  $$fix = \bigsqcup f^i(\bot, \bot, \ldots, \bot) \qquad fix\omega = \bigsqcup (\omega \circ f)^i(\bot, \bot, \ldots, \bot)$$

  then $fix \sqsubseteq fix\omega$
- But we also have that

  $$fix \sqsubseteq f(fix\omega) \sqsubseteq fix\omega$$

  so applying f again may improve the result and remain sound!
- This can be iterated arbitrarily many times
  - may diverge, but safe to stop anytime

Backing up



Narrowing in action

```
y = 0;
x = 7;
x = x+1;
while (input) {
  x = 7;
  x = x+1;
  y = y+1;
}
```

$[x \to \bot, y \to \bot]$
$[x \to [7,\infty], y \to [0,1]]$
$[x \to [7,\infty], y \to [0,7]]$
$[x \to [7,\infty], y \to [0,\infty]]$
…
$[x \to [8,8], y \to [0,\infty]]$

$B = \{-\infty, 0, 1, 7, \infty\}$

Correctness of (repeated) narrowing

- $f(\mathit{fix}\omega) \sqsubseteq \omega(f(\mathit{fix}\omega)) = (\omega \circ f)(\mathit{fix}\omega) = \mathit{fix}\omega$
  since ω is extensive
  - by induction we also have, for all i:

    $f^{i+1}(\mathit{fix}\omega) \sqsubseteq f^{i}(\mathit{fix}\omega) \sqsubseteq \mathit{fix}\omega$
  - i.e. $f^{i+1}(\mathit{fix}\omega)$ is at least as precise as $f^{i}(\mathit{fix}\omega)$
- $\mathit{fix} \sqsubseteq \mathit{fix}\omega$ hence $f(\mathit{fix}) = \mathit{fix} \sqsubseteq f(\mathit{fix}\omega)$
  by monotonicity of f
  - by induction we also have, for all i:

    $\mathit{fix} \sqsubseteq f^{i}(\mathit{fix}\omega)$
  - i.e. $f^{i}(\mathit{fix}\omega)$ is a sound approximation of *fix*

More powerful widening

- A *widening* is a function ∇: $L \times L \rightarrow L$ that is extensive in both arguments and satisfies the following property:

  for all increasing chains $z_0 \sqsubseteq z_1 \sqsubseteq \ldots$,

  the sequence $y_0 = z_0, \ldots, y_{i+1} = y_i \nabla z_{i+1}, \ldots$ converges

  (i.e. stabilizes after a finite number of steps)

- Now replace the basic fixed point solver by computing $x_0 = \bot, \ldots, x_{i+1} = x_i \nabla F(x_i), \ldots$ until convergence

More powerful widening for interval analysis

Extrapolates unstable bounds to B:

$\bot \nabla y = y$

$x \nabla \bot = x$

$[a_1, b_1] \nabla [a_2, b_2] =$

$[\text{if } a_1 \leq a_2 \text{ then } a_1 \text{ else } \mathit{max}\{i \in B \mid i \leq a_2\},$

$\text{if } b_2 \leq b_1 \text{ then } b_1 \text{ else } \mathit{min}\{i \in B \mid b_2 \leq i\}]$

The ∇ operator on L is then defined pointwise down to individual intervals

For the small example program, we get the same result as with simple widening plus narrowing (but now without using narrowing)

More powerful narrowing

- Similarly, we can generalize narrowing
- A *narrowing* is a function $\Delta: L^n \times L^n \rightarrow L^n$ such that

  $\forall x,y \in L^n: (y \sqsubseteq x) \Rightarrow (y \sqsubseteq (x \,\Delta\, y) \sqsubseteq x)$

  and

  for all decreasing chains $x_0 \sqsupseteq x_1 \sqsupseteq \ldots$,

  the sequence $y_0 = x_0, \ldots, y_{i+1} = y_i \,\Delta\, x_{i+1}, \ldots$ converges

- After computing the fixed point y_k with widening, continue with $y_{i+1} = y_i \,\Delta\, F(y_i)$ (until convergence or bounded number of iterations)

Numerical Abstract Domains